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Solution of nonlinear Fredholm integro-differential equations using a hybrid of block pulse functions and normalized Bernstein polynomials

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ABSTRACT

A numerical method for solving nonlinear Fredholm integro-differential equations is proposed. The method is based on hybrid function approximations. The properties of a hybrid of block pulse functions and orthonormal Bernstein polynomials are presented and utilized to reduce the problem to the solution of nonlinear algebraic equations. Numerical examples are introduced to illustrate the effectiveness and simplicity of the present method.

1. Introduction

Integro-differential equations are often involved in mathematical formulations of physical phenomena. Fredholm integro-differential equations play an important role in many fields such as economics, biomechanics, control, elasticity, fluid dynamics, heat and mass transfer, oscillation theory, and airfoil theory; see, for example, [1–3] and the references cited therein. Finding numerical solutions for Fredholm integro-differential equations is one of the oldest problems in applied mathematics. Numerous works have been focusing on the development of more advanced and efficient methods for solving integro-differential equations such as the wavelet method [4,5], Walsh function method [6], sinc-collocation method [7], homotopy analysis method [8], differential transform method [9], using hybrid Legendre polynomials and block pulse functions [10], Chebyshev polynomial method [11], and the Bernoulli matrix method [12].

Block pulse functions have been studied and applied extensively as a basic set of functions for signal and function approximations. All these studies and applications show that block pulse functions have definite advantages for solving problems involving integrals and derivatives due to their clearness in expressions and their simplicity in formulations; see [13]. Also, Bernstein polynomials play a prominent role in various areas of mathematics. Many authors have used these polynomials in the solution of integral equations, differential equations, and approximation theory; see, for instance, [14–17].

The purpose of this work is to utilize hybrid functions consisting of a combination of block pulse functions with normalized Bernstein polynomials to obtain a numerical solution of the nonlinear Fredholm integro-differential equation

$$\sum_{i=0}^{s} p_i(x)y^{(i)}(x) = g(x) + \lambda \int_0^1 k(x,t)[y(t)]^q dt, \quad 0 \le x,\ t < 1, \tag{1}$$

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with the conditions

$$y^{(i)}(0) = \alpha_i, \quad 0 \le i \le s-1, \tag{2}$$

where $y^{(i)}(x)$ is the ith derivative of the unknown function that will be determined, $k(x, t)$ is the kernel of the integral, $g(x)$ and $p_i(x)$ are known analytic functions, q is a positive integer, and λ and α_i are suitable constants. The proposed approach for solving this problem uses a small number of bases, and benefits from the orthogonality of block pulse functions and the advantages of orthonormal Bernstein polynomial properties to reduce the nonlinear integro-differential equation to easily solvable nonlinear algebraic equations.

This paper is organized as follows. In the next section, we present a hybrid of Bernstein polynomials and block pulse functions. Also, their useful properties such as function approximation, convergence analysis, operational matrix of product, and operational matrix of differentiation are given. In Section 3, the numerical scheme for the solution of (1) and (2) is described. In Section 4, the proposed method is applied to some nonlinear Fredholm integro-differential equations, and comparisons are made with the existing analytic or numerical solutions that were reported in other published works in literature. Finally, conclusions are given in Section 5.

2. Properties of hybrid functions

2.1. Hybrid of block pulse functions and orthonormal Bernstein polynomials

The Bernstein polynomials of the nth degree are defined on the interval [0, 1] as [16]

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad \text{for } i = 0, 1, 2, \ldots, n, \tag{3}$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$.

There are $(n+1)$$n$th-degree Bernstein polynomials. Using the Gram–Schmidt orthonormalization process on $B_{i,n}(x)$, we obtain a class of orthonormal polynomials from the Bernstein polynomials. We call them orthonormal Bernstein polynomials of degree n, and denote them by $b_{i,n}(x)$, $0 \le i \le n$. For $n = 3$, the four orthonormal Bernstein polynomials are given by

$$b_{0,3}(x) = -\sqrt{7}[x^3 - 3x^2 + 3x - 1], \qquad b_{1,3}(x) = \sqrt{5}[7x^3 - 15x^2 + 9x - 1],$$
$$b_{2,3}(x) = -\sqrt{3}[21x^3 - 33x^2 + 13x - 1], \quad \text{and} \quad b_{3,3}(x) = 35x^3 - 45x^2 + 15x - 1.$$

Hybrid functions $h_{ji}(x)$, $j = 1, 2, \ldots, m$ and $i = 0, 1, \ldots, n$, are defined on the interval [0, 1) as

$$h_{ji}(x) = \begin{cases} \sqrt{m}\, b_{i,n}(mx - j + 1), & \dfrac{j-1}{m} \le x < \dfrac{j}{m}, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where j and n are the order of the block pulse functions and the degree of the orthonormal Bernstein polynomials, respectively.

It is clear that these sets of hybrid functions in Eq. (4) are orthonormal and disjoint.

2.2. Function approximation

A function $y(x) \in L^2[0, 1)$ may be approximated as

$$y(x) \approx \sum_{j=1}^{m} \sum_{i=0}^{n} c_{ji} h_{ji}(x) = \mathbf{C}^T \mathbf{H}(x), \tag{5}$$

where

$$\mathbf{C} = [\mathbf{C}_1^T, \mathbf{C}_2^T, \ldots, \mathbf{C}_j^T, \ldots, \mathbf{C}_m^T]^T, \tag{6}$$
$$\mathbf{C}_j = [c_{j0}, c_{j1}, c_{j2}, \ldots, c_{jn}]^T, \quad j = 1, 2, \ldots, m,$$
$$\mathbf{H}(x) = [\mathbf{H}_1^T(x), \mathbf{H}_2^T(x), \ldots, \mathbf{H}_j^T(x), \ldots, \mathbf{H}_m^T(x)]^T, \tag{7}$$

and $\mathbf{H}_j(x) = [h_{j0}(x), h_{j1}(x), \ldots, h_{jn}(x)]^T$, $j = 1, 2, \ldots, m$. The constant coefficients c_{ji} are $\big(y(x), h_{ji}(x)\big)$, $i = 0, 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $(\cdot,\ \cdot)$ is the standard inner product on $L^2[0, 1)$.

We can also approximate the function $k(x, t) \in L^2([0, 1) \times [0, 1))$ by

$$k(x, t) \approx \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{l=0}^{n} \sum_{r=0}^{n} k_{lr}^{ij} h_{il}(x) h_{jr}(t) = \mathbf{H}^T(x) \mathbf{K} \mathbf{H}(t), \tag{8}$$

where $\mathbf{K} = [\mathbf{k}^{ij}]$ is an $m(n+1) \times m(n+1)$ matrix, such that the elements of the submatrix $\mathbf{k}^{ij}$ are

$$k_{lr}^{ij} = \int_{i-1/m}^{i/m} \int_{j-1/m}^{j/m} k(x,t) h_{i(l-1)}(x) h_{j(r-1)}(t) dx dt, \quad l, r = 1, 2, \ldots, n+1,\ i, j = 1, 2, \ldots, m, \tag{9}$$

utilizing properties of block pulse functions and orthonormal Bernstein polynomials.

2.3. Convergence analysis

In this section, the error bound and convergence are established by the following lemma.

Lemma 2.1. *Suppose that $f \in C^{(n+1)}[0, 1)$ is an $n+1$ times continuously differentiable function such that $f = \sum_{j=1}^{m} f_j$, and let $Y_j = \mathrm{Span}\{h_{j0}(x), h_{j1}(x), \ldots, h_{jn}(x)\}$, $j = 1, 2, \ldots, m$. If $\mathbf{C}_j^T \mathbf{H}_j(x)$ is the best approximation to f_j from Y_j, then $\mathbf{C}^T \mathbf{H}(x)$ approximates f with the following error bound:*

$$\left\| f - \mathbf{C}^T \mathbf{H}(x) \right\|_2 \leq \frac{\gamma}{m^{n+1}(n+1)!\sqrt{2n+3}}, \quad \gamma = \max_{x \in [0.1)} |f^{(n+1)}(x)|. \tag{10}$$

Proof. The Taylor expansion for the function $f_j(x)$ is

$$\tilde{f}_j(x) = f_j\left(\frac{j-1}{m}\right) + f_j'\left(\frac{j-1}{m}\right)\left(x - \frac{j-1}{m}\right) + \cdots + f_j^{(n)}\left(\frac{j-1}{m}\right)\frac{\left(x - \frac{j-1}{m}\right)^n}{n!}, \quad \frac{j-1}{m} \leq x < \frac{j}{m},$$

for which it is known that

$$|f_j(x) - \tilde{f}_j(x)| \leq |f^{(n+1)}(\eta)| \frac{\left(x - \frac{j-1}{m}\right)^{n+1}}{(n+1)!}, \quad \eta \in [j - 1/m, j/m),\ j = 1, 2, \ldots, m. \tag{11}$$

Since $\mathbf{C}_j^T \mathbf{H}_j(x)$ is the best approximation to f_j form Y_j and $\tilde{f}_j \in Y_j$, using (11), we have

$$\begin{aligned}
\left\| f_j - \mathbf{C}_j^T \mathbf{H}_j(x) \right\|_2^2 &\leq \left\| f_j - \tilde{f}_j \right\|^2 = \int_{j-1/m}^{j/m} |f_j(x) - \tilde{f}_j(x)|^2 dx \\
&\leq \int_{j-1/m}^{j/m} \left[\frac{f^{(n+1)}(\eta)(x - j - 1/m)^{n+1}}{(n+1)!} \right]^2 dx \\
&\leq \left[\frac{\gamma}{(n+1)!} \right]^2 \int_{j-1/m}^{j/m} \left(x - \frac{j-1}{m} \right)^{2n+2} dx = \left[\frac{\gamma}{(n+1)!} \right]^2 \frac{1}{m^{2n+3}(2n+3)}.
\end{aligned}$$

Now,

$$\left\| f - \mathbf{C}^T \mathbf{H}(x) \right\|_2^2 \leq \sum_{j=1}^{m} \left\| f_j - \mathbf{C}_j^T \mathbf{H}_j(x) \right\|_2^2 \leq \frac{\gamma^2}{m^{2n+2}[(n+1)!]^2(2n+3)}.$$

By taking the square roots, we have the above bound. □

2.4. The operational matrix of the product

In this section, we present a general formula for finding the $m(n+1) \times m(n+1)$ operational matrix of the product $\tilde{\mathbf{C}}$ whenever

$$\mathbf{C}^T \mathbf{H}(x) \mathbf{H}^T(x) \approx \mathbf{H}^T(x) \tilde{\mathbf{C}}, \tag{12}$$

where

$$\tilde{\mathbf{C}} = \mathrm{diag}\,[\tilde{\mathbf{C}}_1, \tilde{\mathbf{C}}_2, \ldots, \tilde{\mathbf{C}}_j, \ldots, \tilde{\mathbf{C}}_m]. \tag{13}$$

In Eq. (13), $\tilde{\mathbf{C}}_j = [c_{lr}^j]$ are $(n+1) \times (n+1)$ symmetric matrices depending on n, where

$$c_{lr}^j = \int_{j-1/m}^{j/m} \left(h_{j(l-1)}(x) h_{j(r-1)}(x) \sum_{i=0}^{n} c_{ji} h_{ji}(x) \right) dx, \quad l, r = 1, 2, \ldots, n+1. \tag{14}$$

Furthermore, the integration of the cross-product of two hybrid functions vectors is

$$\int_0^1 \mathbf{H}(x) \mathbf{H}^T(x) dx = \mathbf{I}, \tag{15}$$

where $\mathbf{I}$ is the $m(n+1)$ identity matrix.

2.5. The operational matrix of differentiation

The operational matrix of the derivative of the hybrid function vector $\mathbf{H}(x)$ is defined by

$$\frac{d}{dx}\mathbf{H}(x) = \mathbf{D}\mathbf{H}(x), \tag{16}$$

where $\mathbf{D}$ is the $m(n+1) \times m(n+1)$ operational matrix of the derivative, given as

$$\mathbf{H}(x) = [\mathbf{H}_1^T(x), \mathbf{H}_2^T(x), \dots, \mathbf{H}_j^T(x), \dots, \mathbf{H}_m^T(x)]^T = \tilde{\mathbf{A}}\tilde{\mathbf{T}}(x),$$

where $\tilde{\mathbf{A}} = \mathrm{diag}\,[\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_j, \dots, \mathbf{A}_m]$ is the $m(n+1) \times m(n+1)$ coefficient matrix of the $(n+1) \times (n+1)$ coefficient submatrix $\mathbf{A}_j$, and $\tilde{\mathbf{T}}(x) = [\mathbf{t}_1(x), \mathbf{t}_2(x), \dots, \mathbf{t}_j(x), \dots, \mathbf{t}_m(x)]^T$ is the $m(n+1)$ vector with $\mathbf{t}_j(x) = [1, x, x^2, \dots, x^n]^T$, such that $\mathbf{H}_j(x) = \mathbf{A}_j\mathbf{t}_j(x)$. Now

$$\frac{d}{dx}\mathbf{H}(x) = \tilde{\mathbf{A}}\tilde{\mathbf{Q}}\tilde{\mathbf{T}}(x) = \tilde{\mathbf{A}}\tilde{\mathbf{Q}}\tilde{\mathbf{A}}^{-1}\mathbf{H}(x),$$

where $\tilde{\mathbf{Q}} = \mathrm{diag}\,[\mathbf{Q}, \mathbf{Q}, \dots, \mathbf{Q}, \dots, \mathbf{Q}]$ is the $m(n+1) \times m(n+1)$ matrix of the $(n+1) \times (n+1)$ submatrix $\mathbf{Q}$, such that

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n & 0 \end{bmatrix}.$$

Hence,

$$\mathbf{D} = \tilde{\mathbf{A}}\tilde{\mathbf{Q}}\tilde{\mathbf{A}}^{-1}. \tag{17}$$

In general, we obtain

$$\frac{d^k}{dx^k}\mathbf{H}(x) = \mathbf{D}^k\mathbf{H}(x), \quad k = 1, 2, 3, \dots. \tag{18}$$

3. Outline of the solution method

This section presents the derivation of the method for solving the sth-order nonlinear Fredholm integro-differential equation (1) with the initial conditions (2).

Step 1: The functions $y^{(i)}(x)$, $i = 0, 1, 2, \dots, s$ are approximated by

$$y^{(i)}(x) = \mathbf{C}^T(\mathbf{H}(x))^{(i)} = \mathbf{C}^T\mathbf{D}^i\mathbf{H}(x), \quad i = 0, 1, 2, \dots, s, \tag{19}$$

where $\mathbf{D}$ is given by (17).

Step 2: The function $k(x, t)$ is approximated by (8).

Step 3: In this step, we present a general formula for approximating $y^q(x)$. By using Eqs. (5) and (12), we obtain

$$y^2(x) = [\mathbf{C}^T\mathbf{H}(x)]^2 \quad = \mathbf{C}^T\mathbf{H}(x)\mathbf{H}^T(x)\mathbf{C} = \mathbf{H}^T(x)\tilde{\mathbf{C}}\mathbf{C},$$
$$y^3(x) = \mathbf{C}^T\mathbf{H}(x)[\mathbf{C}^T\mathbf{H}(x)]^2 = \mathbf{C}^T\mathbf{H}(x)\mathbf{H}^T(x)\tilde{\mathbf{C}}\mathbf{C} = \mathbf{H}^T(x)\tilde{\mathbf{C}}\tilde{\mathbf{C}}\mathbf{C} = \mathbf{H}^T(x)(\tilde{\mathbf{C}})^2\mathbf{C},$$

and so by use of induction $y^q(x)$ will be approximated as

$$y^q(x) = \mathbf{H}^T(x)(\tilde{\mathbf{C}})^{q-1}\mathbf{C}. \tag{20}$$

Step 4: Approximate the functions $g(x)$ and $p_i(x)$ by

$$g(x) \approx \mathbf{G}^T\mathbf{H}(x), \tag{21}$$

and

$$p_i(x) \approx \mathbf{P}_i^T H(x), \quad i = 0, 1, 2, \dots, s, \tag{22}$$

where $\mathbf{G}$ and $\mathbf{P}_i$ are constant coefficient vectors which are defined similarly to Eq. (5).

Now, using Eqs. (19)–(22) and (8) to substitute into Eq. (1), we obtain

$$\sum_{i=0}^{s} \mathbf{P}_i^T\mathbf{H}(x)\mathbf{H}^T(x)(\mathbf{D}^i)^T\mathbf{C} = \mathbf{H}^T(x)\mathbf{G} + \lambda\int_0^1 \mathbf{H}^T(x)\mathbf{K}\mathbf{H}(t)\mathbf{H}^T(t)(\tilde{\mathbf{C}})^{q-1}\mathbf{C}dt. \tag{23}$$

Utilizing Eqs. (12) and (15), we obtain

$$\sum_{i=0}^{s} \mathbf{H}^T(x)\tilde{\mathbf{P}}_i(\mathbf{D}^i)^T\mathbf{C} = \mathbf{H}^T(x)\mathbf{G} + \lambda\mathbf{H}^T(x)\mathbf{K}(\tilde{\mathbf{C}})^{q-1}\mathbf{C}, \tag{24}$$

and hence we get

$$\sum_{i=0}^{s} \tilde{\mathbf{P}}_i(\mathbf{D}^i)^T\mathbf{C} - \lambda\mathbf{K}(\tilde{\mathbf{C}})^{q-1}\mathbf{C} = \mathbf{G}. \tag{25}$$

The matrix equation (25) gives a system of $m(n+1)$ nonlinear algebraic equations which can be solved utilizing the initial condition for the elements of $\mathbf{C}$. Once $\mathbf{C}$ is known, $y(x)$ can be constructed by using Eq. (5).

4. Applications and numerical results

In this section, numerical results of some examples are presented to validate the accuracy, applicability, and convergence of the proposed method. The absolute difference errors of this method are compared with those of the existing methods reported in the literature [5,6,17,18]. The computations associated with these examples were performed using Matlab 9.0.

Example 1. Consider the first-order nonlinear Fredholm integro-differential equation [17,18]

$$y'(x) = 1 - \frac{1}{3}x + \int_0^1 x y^2(t)\,dt, \quad 0 \le x < 1, \tag{26}$$

with the initial condition

$$y(0) = 0. \tag{27}$$

In this example, we have $p_0 = 0$, $p_1 = 1$, $g(x) = 1 - \frac{1}{3}x$, $\lambda = 1$, $k(x,t) = x$, and $q = 2$.

The matrix equation (25) for this example is

$$\tilde{\mathbf{P}}_1\mathbf{D}^T\mathbf{C} - \mathbf{K}(\tilde{\mathbf{C}})\mathbf{C} = \mathbf{G}, \tag{28}$$

where, for $n = 1$ and $m = 2$, we have

$$\tilde{\mathbf{P}}_1 = \mathbf{I}, \qquad \mathbf{D}^T = \begin{bmatrix} -3 & 3\sqrt{3} & 0 & 0 \\ -\sqrt{3} & 3 & 0 & 0 \\ 0 & 0 & -3 & 3\sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} c_{10} \\ c_{11} \\ c_{20} \\ c_{21} \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} 1/16 & \sqrt{3}/48 & 1/16 & \sqrt{3}/48 \\ \sqrt{3}/16 & 1/16 & \sqrt{3}/16 & 1/16 \\ 1/4 & \sqrt{3}/12 & 1/4 & \sqrt{3}/12 \\ \sqrt{3}/8 & 1/8 & \sqrt{3}/8 & 1/8 \end{bmatrix},$$

$$\tilde{\mathbf{C}} = \frac{1}{4}\begin{bmatrix} 3\sqrt{6}c_{10} - \sqrt{2}c_{11} & -\sqrt{2}c_{10} + \sqrt{6}c_{11} & 0 & 0 \\ -\sqrt{2}c_{10} + \sqrt{6}c_{11} & \sqrt{6}c_{10} + 5\sqrt{2}c_{11} & 0 & 0 \\ 0 & 0 & 3\sqrt{6}c_{20} - \sqrt{2}c_{21} & -\sqrt{2}c_{20} + \sqrt{6}c_{21} \\ 0 & 0 & -\sqrt{2}c_{20} + \sqrt{6}c_{21} & \sqrt{6}c_{20} + 5\sqrt{2}c_{21} \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} 17\sqrt{6}/72 \\ 5\sqrt{2}/24 \\ 7\sqrt{6}/36 \\ \sqrt{2}/6 \end{bmatrix}.$$

Eq. (28) gives a system of nonlinear algebraic equations that can be solved utilizing the initial condition (27); i.e., $\sqrt{6}c_{10} - \sqrt{2}c_{11} = 0$. We obtain

$$c_{10} = \sqrt{6}/24, \qquad c_{11} = \sqrt{2}/8, \qquad c_{20} = \sqrt{6}/6, \quad \text{and} \quad c_{21} = \sqrt{2}/4.$$

Substituting these values into (5), the result will be $y(x) = x$, which is the exact solution. It is noted that the result gives the exact solution as in [17], while in [18] using the sinc method the maximum absolute error is 1.5217E−03.

Example 2. Consider the first-order nonlinear Fredholm integro-differential equation [6,17]

$$xy'(x) - y(x) = -\frac{1}{6} + \frac{4}{5}x^2 + \int_0^1 (x^2 + t)y^2(t)dt, \quad 0 \le x < 1, \tag{29}$$

with the initial condition

$$y(0) = 0. \tag{30}$$

In this example, we have $p_0 = -1$, $p_1 = x$, $g(x) = -\frac{1}{6} + \frac{4}{5}x^2$, $\lambda = 1$, $k(x,t) = x^2 + t$, and $q = 2$.

The matrix equation (25) for this example is

$$(\tilde{\mathbf{P}}_0 + \tilde{\mathbf{P}}_1 \mathbf{D}^T)\mathbf{C} - \mathbf{K}(\tilde{\mathbf{C}})\mathbf{C} = \mathbf{G}, \tag{31}$$

where for $n = 2$ and $m = 2$ we have

$$\tilde{\mathbf{P}}_0 = -\mathbf{I}, \qquad \tilde{\mathbf{P}}_1 = \begin{bmatrix} 1/12 & \sqrt{15}/60 & -\sqrt{5}/120 & 0 & 0 & 0 \\ \sqrt{15}/60 & 1/4 & \sqrt{3}/24 & 0 & 0 & 0 \\ -\sqrt{5}/120 & \sqrt{3}/24 & 5/12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7/12 & \sqrt{15}/60 & -\sqrt{5}/120 \\ 0 & 0 & 0 & \sqrt{15}/60 & 3/4 & \sqrt{3}/24 \\ 0 & 0 & 0 & -\sqrt{5}/120 & \sqrt{3}/24 & 11/12 \end{bmatrix}$$

$$\mathbf{D}^T = \begin{bmatrix} -5 & 7\sqrt{15}/3 & -2\sqrt{5} & 0 & 0 & 0 \\ -\sqrt{15}/3 & -3 & 14\sqrt{3}/3 & 0 & 0 & 0 \\ 0 & -8\sqrt{3}/3 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 7\sqrt{15}/3 & -2\sqrt{5} \\ 0 & 0 & 0 & -\sqrt{15}/3 & -3 & 14\sqrt{3}/3 \\ 0 & 0 & 0 & 0 & -8\sqrt{3}/3 & 8 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} c_{10} \\ c_{11} \\ c_{12} \\ c_{20} \\ c_{21} \\ c_{22} \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} -11\sqrt{10}/450 \\ -\sqrt{6}/90 \\ \sqrt{2}/180 \\ 23\sqrt{10}/900 \\ 13\sqrt{6}/180 \\ 19\sqrt{2}/180 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 1/24 & \sqrt{15}/45 & 7\sqrt{5}/240 & 13/72 & \sqrt{15}/20 & 41\sqrt{5}/720 \\ \sqrt{15}/72 & 1/12 & 5\sqrt{3}/144 & \sqrt{15}/24 & 1/16 & \sqrt{3}/16 \\ \sqrt{5}/48 & 5\sqrt{3}/144 & 1/24 & 7\sqrt{5}/144 & \sqrt{3}/16 & 5/72 \\ 7/48 & 31\sqrt{15}/720 & \sqrt{15}/20 & 41/144 & 17\sqrt{15}/240 & 7\sqrt{5}/90 \\ 7\sqrt{15}/144 & 3/16 & 5\sqrt{3}/72 & 11\sqrt{15}/144 & 13/48 & 7\sqrt{3}/72 \\ \sqrt{5}/16 & 11\sqrt{3}/144 & 1/12 & 13\sqrt{5}/144 & 5\sqrt{3}/48 & 1/9 \end{bmatrix}, \qquad \tilde{\mathbf{C}} = \begin{bmatrix} \tilde{\mathbf{c}}_1 & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{c}}_2 \end{bmatrix},$$

$$\tilde{\mathbf{c}}_j = \begin{bmatrix} \frac{5\sqrt{10}}{7}c_{j0} - \frac{5\sqrt{6}}{21}c_{j1} + \frac{\sqrt{2}}{7}c_{j2} & -\frac{5\sqrt{6}}{21}c_{j0} + \frac{11\sqrt{10}}{35}c_{j1} - \frac{8\sqrt{30}}{105}c_{j2} & \frac{\sqrt{2}}{7}c_{j0} - \frac{8\sqrt{30}}{105}c_{j1} + \frac{3\sqrt{10}}{35}c_{j2} \\ -\frac{5\sqrt{6}}{21}c_{j0} + \frac{11\sqrt{10}}{35}c_{j1} - \frac{8\sqrt{30}}{105}c_{j2} & \frac{11\sqrt{10}}{35}c_{j0} + \frac{3\sqrt{6}}{7}c_{j1} + \frac{\sqrt{2}}{7}c_{j2} & -\frac{8\sqrt{30}}{105}c_{j0} + \frac{\sqrt{2}}{7}c_{j1} + \frac{5\sqrt{6}}{21}c_{j2} \\ \frac{\sqrt{2}}{7}c_{j0} - \frac{8\sqrt{30}}{105}c_{j1} + \frac{3\sqrt{10}}{35}c_{j2} & -\frac{8\sqrt{30}}{105}c_{j0} + \frac{\sqrt{2}}{7}c_{j1} + \frac{5\sqrt{6}}{21}c_{j2} & \frac{3\sqrt{10}}{35}c_{j0} + \frac{5\sqrt{6}}{21}c_{j1} + \frac{13\sqrt{2}}{7}c_{j2} \end{bmatrix},$$

$j = 1, 2.$

Eq. (31) gives a system of nonlinear algebraic equations that can be solved utilizing the initial condition (30); i.e., $\sqrt{10}c_{10} - \sqrt{6}c_{11} + \sqrt{2}c_{12} = 0$. We obtain

$$c_{10} = \sqrt{10}/240, \quad c_{11} = \sqrt{6}/48, \quad c_{12} = \sqrt{2}/24, \quad c_{20} = \sqrt{10}/15, \quad c_{21} = \sqrt{6}/8, \quad \text{and} \quad c_{22} = \sqrt{2}/6.$$

Substituting these values into (5), the result will be $y(x) = x^2$, which is the exact solution. It is noted that the result gives the exact solution as in [17], while in [6] approximate solution is obtained with maximum absolute error 1.0000E−10.

Example 3. Consider the second order nonlinear Fredholm integro-differential equation [17]

$$y''(x) + xy'(x) - xy(x) = e^x - \sin x + \int_0^1 \sin x.\, e^{-2t} y^2(t)dt, \quad 0 \le x < 1, \tag{32}$$

with the initial conditions

$$y(0) = y'(0) = 1. \tag{33}$$

The exact solution is $y(x) = e^x$. We solve this example by using the proposed method with $n = 2$ and $m = 30$ and with $n = 3$ and $m = 30$. A comparison between the proposed method and the methods in [17] is shown in Table 1. It is clear from this table that the results obtained by the proposed method, using a small number of bases, are very promising and superior to those of [17].

Table 1
Numerical comparison of absolute difference errors for Example 3.

x	Method of [17]	The proposed method	
	$n = 7$	$n = 2, m = 30$	$n = 3, m = 30$
0.0	3.2038E−009	3.1309E−007	4.0173E−010
0.2	7.1841E−010	3.8241E−007	4.9068E−010
0.4	1.4151E−010	4.6707E−007	5.9932E−010
0.6	4.0671E−011	5.7048E−007	7.3201E−010
0.8	9.1044E−010	6.9679E−007	8.9407E−010
1.0	3.7002E−009	8.2709E−007	1.4907E−010

Table 2
Numerical comparison of absolute difference errors for Example 4.

x	Method of [5]	Method of [17]	The proposed method	
	No. of collocation points $N = 128$	$n = 7$	$n = 3, m = 35$	$n = 4, m = 15$
0.125	3.7591E−007	2.4509E−010	5.5200E−011	1.6710E−011
0.250	6.6413E−007	1.0202E−010	8.9982E−011	3.9705E−012
0.375	8.6917E−007	1.6139E−010	9.4606E−011	1.2126E−011
0.500	1.0020E−006	3.2362E−010	9.2457E−011	1.8312E−012
0.625	1.0757E−006	1.9197E−010	7.4991E−011	8.1299E−012
0.750	1.1029E−006	6.6120E−011	4.9442E−011	7.7237E−012
0.875	1.0944E−006	2.2417E−010	2.6083E−011	2.5547E−012

Table 3
Maximum absolute errors for different values of n and m for Example 4.

n	m							
	1	5	10	15	20	25	30	35
0	5.7735E−01	1.1547E−01	5.7735E−02	3.8490E−02	2.8868E−02	2.3094E−02	1.9245E−02	1.6496E−02
1	2.2361E−01	8.9443E−03	2.2361E−03	9.9381E−04	5.5902E−04	3.5777E−04	2.4845E−04	1.8254E−04
2	6.2994E−02	5.0395E−04	6.2994E−05	1.8665E−05	7.8743E−06	4.0316E−06	2.3331E−06	1.4693E−06
3	1.3889E−02	2.2222E−05	1.3889E−06	2.7435E−07	8.6806E−08	3.5556E−08	1.7147E−08	9.2554E−09
4	2.5126E−03	8.0403E−07	2.5126E−08	3.3088E−09	7.8519E−10	2.5729E−10	1.0340E−10	4.7839E−11
5	3.8521E−04	2.4653E−08	3.8521E−10	3.3818E−11	6.0189E−12	1.5778E−12	5.2841E−13	2.0955E−13
6	5.1230E−05	6.5574E−10	5.1230E−12	2.9984E−13	4.0023E−14	8.3935E−15	2.3425E−15	7.9625E−16

Example 4. Consider the following nonlinear Fredholm integro-differential equation [5,17]

$$y'(x) + y(x) = \frac{1}{2}(e^{-2} - 1) + \int_0^1 y^2(t)\,dt, \quad 0 \le x < 1, \tag{34}$$

with the initial condition

$$y(0) = 1. \tag{35}$$

The exact solution of this problem is $y(x) = e^{-x}$. In Table 2 we have compared the absolute difference errors of the proposed method with those of the collocation method based on Haar wavelets in [5] and the method in [17].

The maximum absolute errors of Example 4 for some different values of n and m are shown in Table 3. As is seen from Table 3, for a certain value of n, as m increases, the accuracy increases, and for a certain value of m, as n increases, the accuracy increases as well. When $m = 1$ the numerical solution obtained is based on orthonormal Bernstein polynomials only, while when $n = 0$ the numerical solution obtained is based on block pulse functions only.

Example 5. Consider the first-order nonlinear Fredholm integro-differential equation [17,18]

$$y'(x) = e^x - \frac{1}{5}e^{-x^2}(e^5 - 1) + \int_0^1 e^{2t-x^2} y^3(t)\,dt, \quad 0 \le x < 1, \tag{36}$$

with the initial condition

$$y(0) = 1. \tag{37}$$

The absolute difference errors of the proposed method for $n = 4$ and $m = 35$ and for $n = 5$ and $m = 20, 35$, and the absolute difference errors of the method in [17] are displayed in Table 4. The results obtained by the proposed method using four basis functions are better than those of [17] using nine basis functions. Of course, as n increases, the accuracy improves. Also, the maximum absolute error in [18] using the sinc method is 3.7259E−03. The exact solution of this problem is $y(x) = e^{-x}$.

Table 4
Numerical comparison of absolute difference errors for Example 5.

x	Method of [17]	The proposed method		
	$n = 9$	$n = 4, m = 35$	$n = 5, m = 20$	$n = 5, m = 35$
0.0	2.4740E−012	6.3793E−013	2.3981E−014	3.3307E−016
0.2	1.9780E−012	7.7915E−013	2.9310E−014	4.4409E−016
0.4	2.5981E−012	9.5146E−013	3.5749E−014	4.4507E−016
0.6	3.8940E−012	1.1622E−012	4.3743E−014	6.6613E−016
0.8	5.7709E−012	1.4198E−012	5.3735E−014	8.8818E−016
1.0	3.3360E−012	1.6898E−012	6.2617E−014	1.3232E−015

5. Conclusion

In this work, we have presented a numerical method for solving nonlinear Fredholm integro-differential equations based on a hybrid of block pulse functions and normalized Bernstein polynomials. One of the most important properties of this method is obtaining the analytical solutions if the equation has an exact solution that is a polynomial function. Another considerable advantage is that this method has high relative accuracy for small numbers of bases n. The matrices $\mathbf{K}$, $\tilde{\mathbf{C}}$, and $\mathbf{D}$ in (8), (12) and (17), respectively, have large numbers of zero elements, and they are sparse; hence the present method is very attractive, and it reduces the CPU time and amount of computer memory required. Moreover, satisfactory results of illustrative examples with respect to several other methods (e.g., the Haar wavelet method, Walsh function method, Bernstein polynomial method, and sinc collocation method) have been included to demonstrate the validity and applicability of the proposed method.

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